

An orthogonal polynomial coefficient formula for the Hankel transform

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Abstract

We give an explicit formula for the Hankel transform of a regular sequence in terms of the coefficients of the associated orthogonal polynomials and the sequence itself. We apply this formula to some sequences of combinatorial interest, deriving interesting combinatorial identities by this means. Further insight is also gained into the structure of the Hankel transform of a sequence.

Let μ_n denote a sequence of numbers, with $\mu_0 = 1$. The Hankel transform [16] of this sequence is the sequence h_n of determinants

$$h_n = |\mu_{i+j}|_{0 \leq i,j \leq n}.$$

We shall assume that the sequence μ_n is *regular* (or “Catalan-like”), that is, $h_n \neq 0$ for any n . Much research [4, 6, 9, 10, 13, 19, 20, 21, 22] has focused on finding closed form expressions for the Hankel transforms of important sequences. Some well-known sequences have Hankel transforms with simple formulas (such as our examples below), but in general Hankel transforms can have quite complex forms [9], or be totally resistant to current techniques of elucidation. The sequences we shall use in this note are susceptible to the following classical approach. If [14, 15] μ_n has a generating function $g(x)$ expressible in the form

$$g(x) = \frac{\mu_0}{1 - \alpha_0 x - \frac{\lambda_1 x^2}{1 - \alpha_1 x - \frac{\lambda_2 x^2}{1 - \alpha_2 x - \frac{\lambda_3 x^2}{1 - \alpha_3 x - \dots}}}}$$

then we have [14]

$$h_n = \mu_0^{n+1} \lambda_1^n \lambda_2^{n-1} \cdots \lambda_{n-1}^2 \lambda_n = \mu_0^{n+1} \prod_{k=1}^n \lambda_k^{n+1-k}. \quad (1)$$

We can associate a sequence of monic orthogonal polynomials [5, 11, 25] $P_n(x)$ with the sequence μ_n . The μ_n then correspond to the moments of this family, for the linear functional \mathcal{L} defined by

$$\mathcal{L}(x^n) = \mu_n.$$

The family $\{P_n(x)\}$ satisfies a three term recurrence relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \lambda_n P_{n-1}(x), \quad n \geq 0,$$

where $\lambda_0 P_{-1}(x) = 0$ and $P_1(x) = 1$. We can construct the $P_n(x)$ explicitly as follows [12]. Let

$$\Delta_{i,n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_i \\ \mu_1 & \mu_2 & \cdots & \mu_{i+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{i-1} & \mu_i & \cdots & \mu_{2i-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{n+1} \end{vmatrix}, \quad D_n(x) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_i \\ \mu_1 & \mu_2 & \cdots & \mu_{i+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{i-1} & \mu_i & \cdots & \mu_{2i-1} \\ 1 & x & \cdots & x^n \end{vmatrix},$$

and let

$$D_n = \Delta_{n,n}.$$

Then

$$P_n(x) = \frac{D_n(x)}{D_{n-1}}$$

is the monic orthogonal sequence for \mathcal{L} . The orthogonality of the polynomials is equivalent to

$$\mathcal{L}(P_m P_n) = \lambda_1 \lambda_2 \cdots \lambda_n \delta_{m,n}.$$

In particular,

$$\mathcal{L}(P_n^2) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

However, we know that [20]

$$h_n = \prod_{k=0}^n \lambda_1 \lambda_2 \cdots \lambda_k.$$

Thus we have that

$$h_n = \prod_{k=0}^n \mathcal{L}(P_k^2).$$

We wish now to evaluate $\mathcal{L}(P_k^2)$. To do this, we let $(a_{n,k})$ be the matrix of coefficients of the polynomials $\{P_n(x)\}$. Since the degree of $P_n(x)$ is n , this is a lower triangular matrix, with 1's on the diagonal since the polynomials are monic. It is thus invertible (and if its elements are integers, so are those of the inverse), and we know that [1] the first column of its inverse is composed of the μ_n . Using the Cauchy formula for the coefficients of the product of two polynomials, along with the fact that \mathcal{L} operates on polynomials as follows:

$$\mathcal{L}\left(\sum_{j=0}^m a_j x^j\right) = \sum_{j=0}^m a_j \mu_j,$$

we obtain the following formula:

$$\mathcal{L}(P_k^2) = \sum_{i=0}^{2k} \left(\sum_{j=0}^i a_{k,j} a_{k,i-j} \right) \mu_i. \quad (2)$$

Thus we obtain the following formula for the Hankel transform h_n in terms of the elements of the coefficient array for the associated orthogonal polynomials:

$$h_n = \prod_{k=0}^n \sum_{i=0}^{2k} \left(\sum_{j=0}^i a_{k,j} a_{k,i-j} \right) \mu_i. \quad (3)$$

We state this as a proposition.

Proposition 1. *Let μ_n be a regular sequence, whose associated family of orthogonal polynomials $\{P_n(x)\}_{n \geq 0}$ has coefficient array $(a_{n,k})_{n,k \geq 0}$. Then the Hankel transform h_n of μ_n is given by*

$$h_n = \prod_{k=0}^n \sum_{i=0}^{2k} \left(\sum_{j=0}^i a_{k,j} a_{k,i-j} \right) \mu_i.$$

Note also that the elements of this product, namely $\lambda_1 \lambda_2 \cdots \lambda_k$, are precisely the elements of D where [17, 18, 26] the Hankel matrix H of μ_n has the LDU decomposition

$$H_\mu = (\mu_{i+j}) = LDL^T,$$

where L is the inverse of $(a_{i,j})$. Thus the elements of D are given by

$$\sum_{i=0}^{2n} \left(\sum_{j=0}^i a_{n,j} a_{n,i-j} \right) \mu_i, \quad n = 0, 1, 2, \dots$$

We state this as a corollary to the proposition.

Corollary 2. *Let μ_n be a regular sequence, with Hankel matrix $H = (\mu_{i+j})_{i,j \geq 0}$. Then*

$$H = LDL^T,$$

where the elements d_n of the diagonal matrix D are expressible as

$$d_k = \sum_{i=0}^{2k} \left(\sum_{j=0}^i a_{k,j} a_{k,i-j} \right) \mu_i.$$

1 Examples

In this section, we give some examples, concentrating on “simple” sequences of combinatorial significance whose Hankel transforms are well known. In each case the coefficient array of the associated family of monic orthogonal polynomials is a Riordan array [7, 23, 24]. These examples further explore links [1, 2, 3] between Riordan arrays, orthogonal polynomials and Hankel transforms. For those not familiar with Riordan arrays, we recall that the *Riordan group* [23, 24], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1x + g_2x^2 + \cdots$ and $f(x) = f_1x + f_2x^2 + \cdots$ where $f_1 \neq 0$ [24]. We assume in addition that $f_1 = 1$ in what follows. The associated matrix

is the matrix whose i -th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where \bar{f} is the compositional inverse of f (defined by $f(\bar{f}(x)) = \bar{f}(f(x)) = x$).

Example 3. The binomial matrix (Pascal's triangle) which is the matrix with general term $\binom{n}{k}$ is the Riordan array

$$\mathbf{B} = \left(\frac{1}{1-x}, \frac{x}{1-x} \right).$$

It has inverse $\mathbf{B}^{-1} = ((-1)^{n-k} \binom{n}{k})$ given by

$$\mathbf{B}^{-1} = \left(\frac{1}{1-x}, \frac{x}{1-x} \right).$$

The *exponential Riordan group* [7, 8], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1x + g_2x^2 + \dots$ and $f(x) = f_1x + f_2x^2 + \dots$ where $g_0 \neq 0$ and $f_1 \neq 0$. In what follows, we shall assume

$$g_0 = f_1 = 1.$$

The associated matrix is the matrix whose i -th column has exponential generating function $g(x)f(x)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by $[g, f]$. The group law is given by

$$[g, f] \cdot [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where \bar{f} is the compositional inverse of f .

Example 4. The binomial matrix $\mathbf{B} = \left(\binom{n}{k} \right)$ is also an element of the exponential Riordan group. We have

$$\mathbf{B} = [e^x, x] \quad \text{and} \quad \mathbf{B}^{-1} = [e^{-x}, x].$$

We are now in a position to apply our proposition to some sequences of combinatorial interest.

Example 5. The Catalan numbers [A000108](#). It is well known that for the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ we have $h_n = 1$ for all n . This can be seen since the generating function $g(x) = \frac{1-\sqrt{1-4x}}{2x}$ of the Catalan numbers can be expressed as

$$g(x) = \frac{1}{1-x - \frac{x^2}{1-2x - \frac{x^2}{1-2x - \frac{x^2}{1-\dots}}}}.$$

The corresponding monic orthogonal polynomials

$$P_n(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n+k}{2k} x^k$$

have coefficient array $\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$. We have

$$P_{n+1}(x) = (x-2)P_n(x) - P_{n-1}(x), \quad P_0(x) = 1, \quad P_1(x) = x-1.$$

Equation (3) takes the form

$$\prod_{k=0}^n \sum_{i=0}^{2k} \sum_{j=0}^i (-1)^i \binom{k+j}{2j} \binom{k+i-j}{2(i-j)} C_i = 1.$$

In fact in this case, since $\lambda_i = 1$, we obtain the identity

$$\sum_{i=0}^{2k} \sum_{j=0}^i (-1)^i \binom{k+j}{2j} \binom{k+i-j}{2(i-j)} C_i = 1.$$

Example 6. The central binomial numbers [A000984](#). The Hankel transform of the central binomial numbers $\binom{2n}{n}$ is given by $h_n = 2^n$. This can be seen since the generating function $g(x) = \frac{1}{\sqrt{1-4x}}$ of the central binomial numbers can be expressed as

$$g(x) = \frac{1}{1-2x - \frac{2x^2}{1-2x - \frac{x^2}{1-2x - \frac{x^2}{1-\dots}}}}.$$

The corresponding orthogonal polynomials $P_n(x)$ have coefficient array $\left(\frac{1-x}{1+x}, \frac{x}{(1+x)^2}\right)$. In this case, we have

$$P_{n+1}(x) = (x-2)P_n(x) - P_{n-1}(x), \quad P_0(x) = 1, \quad P_1(x) = x-2.$$

The general term $a_{n,k}$ of the Riordan array $\left(\frac{1-x}{1+x}, \frac{x}{(1+x)^2}\right)$ is given by

$$a_{n,k} = \left(\binom{n+k}{n-k} + \binom{n+k-1}{n-k-1} \right) (-1)^{n-k} = \binom{n+k}{2k} \frac{2n+0^{n+k}}{n+k+0^{n+k}} (-1)^{n-k}.$$

Here we have used the notation 0^n to denote the sequence with term 1, 0, 0, 0, ... and generating function 1. Thus $0^n = \delta_{0,n} = [n=0]$. With this value for $a_{n,k}$, we then obtain

$$\prod_{k=0}^n \sum_{i=0}^{2k} \sum_{j=0}^i a_{k,j} a_{k,i-j} \binom{2i}{i} = 2^n.$$

Now $\lambda_1 = 2$, $\lambda_i = 1$ for $i > 1$, and so we also have

$$\sum_{i=0}^{2k} \left(\sum_{j=0}^i a_{k,j} a_{k,i-j} \right) \binom{2i}{i} = 2 - 0^k.$$

Thus we have, for instance,

$$\sum_{i=0}^{2k} \sum_{j=0}^i (-1)^i \binom{k+j}{2j} \binom{k+i-j}{2(i-j)} \frac{2k+0^{k+j}}{k+j+0^{k+j}} \frac{2k+0^{k+i-j}}{k+j+0^{k+i-j}} \binom{2i}{i} = 2 - 0^k.$$

Example 7. The large Schröder numbers [A006318](#). The large Schröder numbers $S_n = \sum_{k=0}^n \binom{n+k}{2k} C_k$ have Hankel transform $h_n = 2^{\binom{n+1}{2}}$. This can be seen since the generating function $g(x)$ of the large Schröder numbers can be expressed as

$$g(x) = \frac{1}{1 - 2x - \frac{2x^2}{1 - 3x - \frac{2x^2}{1 - 3x - \frac{2x^2}{1 - \dots}}}}.$$

The corresponding monic orthogonal polynomials $P_n(x)$ satisfy

$$P_{n+1}(x) = (x-3)P_n(x) - 2P_{n-1}(x), \quad P_0(x) = 1, \quad P_1(x) = x-2,$$

with coefficient array equal to the Riordan array

$$\left(\frac{1}{1+2x}, \frac{x}{1+3x+2x^2} \right).$$

For this array, we have

$$a_{n,k} = (-1)^{n-k} \sum_{j=0}^n \binom{n-j}{k} \binom{n+k}{j}.$$

With this value for $a_{n,k}$, we then have

$$h_n = 2^{\binom{n+1}{2}} = \prod_{k=0}^n \sum_{i=0}^{2k} \left(\sum_{j=0}^i a_{k,j} a_{k,i-j} \right) S_i,$$

and

$$\sum_{i=0}^{2k} \left(\sum_{j=0}^i a_{k,j} a_{k,i-j} \right) S_i = 2^k.$$

Explicitly, this last equation gives us

$$\sum_{i=0}^{2k} \sum_{j=0}^i \sum_{l=0}^k \binom{k-l}{j} \binom{k+j}{l} \sum_{m=0}^k \binom{k-m}{i-j} \binom{k+i-j}{m} S_i = 2^k.$$

Example 8. The factorial numbers [A000142](#). The factorial numbers $n!$ have Hankel transform $h_n = \prod_{i=0}^n i!^2$. This can be seen since the generating function $g(x)$ of the factorial numbers can be expressed as

$$g(x) = \frac{1}{1 - x - \frac{x^2}{1 - 3x - \frac{4x^2}{1 - 5x - \frac{9x^2}{1 - \dots}}}}.$$

The corresponding monic orthogonal polynomials $P_n(x)$ satisfy

$$P_{n+1}(x) = (x - (2n + 1))P_n(x) - n^2 P_{n-1}(x), \quad P_0(x) = 1, \quad P_1(x) = x - 1,$$

with coefficient array equal to the exponential Riordan array

$$\left[\frac{1}{1+x}, \frac{x}{1+x} \right]$$

with general term $a_{n,k}$ given by

$$a_{n,k} = \binom{n}{k} \frac{n!}{k!} (-1)^{n-k}.$$

With this value for $a_{n,k}$, we then obtain

$$\prod_{k=0}^n \sum_{i=0}^{2k} \left(\sum_{j=0}^i a_{k,j} a_{k,i-j} \right) i! = \prod_{i=0}^n i!^2,$$

and

$$\sum_{i=0}^{2k} \left(\sum_{j=0}^i a_{k,j} a_{k,i-j} \right) i! = k!^2.$$

Writing these expressions explicitly, we get

$$\prod_{k=0}^n \sum_{i=0}^{2k} (-1)^i \binom{k}{j} \binom{k}{i-j} \frac{k!^2}{j!(i-j)!} i! = \prod_{i=0}^n i!^2,$$

and

$$\sum_{i=0}^{2k} \left(\sum_{j=0}^i (-1)^i \binom{k}{j} \binom{k}{i-j} \frac{k!^2}{j!(i-j)!} \right) i! = k!^2.$$

We deduce that

$$\sum_{i=0}^{2k} \left(\sum_{j=0}^i (-1)^i \binom{k}{j} \binom{k}{i-j} \frac{1}{j!(i-j)!} \right) i! = 1,$$

or

$$\sum_{i=0}^{2k} \left(\sum_{j=0}^i (-1)^i \binom{k}{j} \binom{k}{i-j} \binom{i}{j} \right) = 1.$$

Expanding the inner term in i and k as an array, we see that this expression represents the row sums of the array that begins

$$\begin{array}{cccccccccccc} 1 & & & & & & & & & & & \\ 1 & -2 & 2 & & & & & & & & & \\ 1 & -4 & 10 & -12 & 6 & & & & & & & \\ 1 & -6 & 24 & -56 & 78 & -60 & 20 & & & & & \\ 1 & -8 & 44 & -152 & 346 & -520 & 500 & -280 & 70 & & & \\ 1 & -10 & 70 & -320 & 1010 & -2252 & 3560 & -3920 & 2870 & -1260 & 252 & \end{array}$$

Example 9. The derangement (or rencontres) numbers [A000166](#). It is well-known [16] that a sequence and its binomial transform, or its inverse binomial transform, have the same Hankel transform. Thus the derangement numbers [22]

$$d_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k!,$$

which are the inverse binomial transform of the factorial numbers $n!$, have Hankel transform $h_n = \prod_{i=0}^n i!^2$. This may be seen from the continued fraction expansion of their generating function, given by

$$\cfrac{1}{1 - \cfrac{\cfrac{x^2}{1 - 2x - \cfrac{4x^2}{1 - 4x - \cfrac{9x^2}{1 - \dots}}}}{x^2}}.$$

In this case, the coefficient array of the associated orthogonal polynomials is equal to that of the factorial numbers, multiplied on the left by the binomial matrix. We then obtain

$$[e^x, x] \cdot \left[\frac{1}{1+x}, \frac{x}{1+x} \right] = \left[\frac{e^x}{1+x}, \frac{x}{1+x} \right],$$

as the Riordan array expression for the coefficient array of the associated orthogonal polynomials, with general term $a_{n,k}$ given by

$$a_{n,k} = \sum_{j=0}^n \binom{n}{j} \binom{j}{k} \frac{j!}{k!} (-1)^{j-k}.$$

With this value for $a_{n,k}$, we then deduce, for instance, that

$$\sum_{i=0}^{2k} \left(\sum_{j=0}^i a_{k,j} a_{k,i-j} \right) \sum_{l=0}^i \binom{i}{l} (-1)^{i-l} l! = k!^2.$$

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Concerns sequences [A000108](#), [A000142](#), [A000166](#), [A000984](#), [A006318](#).